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The Lorenz System

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Figure 1: Edward Norton Lorenz

1 Introduction

1.1 Historical background

Edward N. Lorenz [1] developed his system of ordinary differential equations in 1963 as part of a physical model,to describe the convention roles in the atmosphere of our planet. The physical backgrounds were developed by the scientists Henri Benard and Lord Rayleigh in about 1900. Investigations from the mathematical side were made in the 60's by Barry Saltzmann and Edward Lorenz. For a longe time it could not be proved that the Lorenz attractor is a strange attractor. Warwick Tucker was the first one,who proved in 1998.

1.2 Overview

The idea, Lorenz had by creating his system was the following: He imagined a planet,whose atmosphere consists of one single fluid particle. The physical circumtances are the same, as on earth. This means, the particle is heated from below and cooled from above. Heating leads to the rising of the particle and then it falls back down because of cooling. This one particle system is improper to predict the weather on our planet.So, Lorenz had to be a little more precise. He looked at a two-dimensional fluid cell that was heated from below and cooled from above. If we want to describe the motion of this fluid cell, we have to deal with a system of differential equations, involving infinitely many variables. This is the place, where Lorenz made a simplifying assumption, that all but three variables remained constant.
Roughly speaking, we have three independent variables left:

1. x = rate of convective overturning
2. y = horizontal temperature variation
3. z = vertical temperature variation

[1] Edward Norton Lorenz was an American mathematician, meteorologist, and a pioneer of chaos theory.

This symplifying assumption led him to the following three-dimensional system of ordinary differential equations:

$$\dot{x} = \sigma(y - x) \tag{1.1}$$

$$\dot{y} = rx - y - xz \tag{1.2}$$

$$\dot{z} = xy - bz \tag{1.3}$$

In this system all parameters are assumed to be positive.
Lorenz used the parameters $\sigma = 10, b = \frac{8}{3}$ and $r = 28$ and initial conditions $P_1 = (0, 2, 0)$ and $P_2 = (0, -2, 0)$ to display the solution curves of his system . Considering the solution curves led him to his discovery.

1.2.1 Definition of convection

Definition:

Generally, convection is the concerted transport of mass and energy within fluids (e.g. liquids and gases)

1.2.2 Derivation of the Lorenz equations

In this section we want to show that the Lorenz system is derived from the so called Navier-Stokes equation for fluid flow and describes thermal energy diffusion. As already mentioned, the Lorenz equations describe the motion of a fluid under conditions of the **Rayleigh-Benard flow:** [2]
An incompressible fluid is contained in a cell which has a higher temperature T_{bot} at the bottom and a lower temperature T_{top} at the top of the cell. It is clear that the temperature difference is given by the formula

$$\Delta T = T_{bot} - T_{top} \tag{1.4}$$

Now, we consider the temperature gradient between the top and bottom plates. If it becomes large enough a small packet of fluid that happens to be moving up a bit experiences an upward buoyant force because of the moving in region with lower temperature.As consequence the density in this region is higher, which means, our fluid packet is less dense than the surroundings around it. That means, our packet looses energy to its environment $\Rightarrow$ If the upward force is strong enough, the packet moves upward more quickly, than its temperature can drop.Then convective currents begin to flow [5].
Now we can try to determine the Lorenz equations. First we have to work with the incompressible Navier-Stokes equation within the Boussinesq approximation. [3] We denote the distance between the top and the bottom of the cell with d.
We have:

$$\nabla \cdot u = 0 \tag{1.5}$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 u + +\alpha g(T - T_{top})\hat{z} \tag{1.6}$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T \tag{1.7}$$

[2] The Rayleigh-Benard flow is a kind of natural convention,occurring in a plane horizontal layer of fluid heated from below, in which the fluid develops a regular pattern of convection cells known as Benard cells.

[3] used for modeling the thermal convection between two parallel horizontal flat plates separated by a distance > 0

1. ρ_0 : density of the fluid at T_0
2. ν : kinematic viscosity of the liquid
3. α : coefficient of thermal expansion
4. κ : thermal diffusivity
5. $\hat{z}$: unit vector in the vertical direction

As usual by solving differential equations, we have to introduce some dimensionless variables:

$$\tilde{x} = \frac{x}{d} \tag{1.8}$$

$$\tilde{u} = \frac{d}{\kappa}u \tag{1.9}$$

$$\tilde{p} = \frac{1}{\rho_0}(\frac{d}{\kappa})^2 p \tag{1.10}$$

$$\tilde{t} = \frac{\kappa}{d^2}t \tag{1.11}$$

We also introduce a new variable Θ for the temperature disturbance. The dimensionless form of it is given by :

$$\tilde{\Theta} = \frac{T - T_{bot}}{T_{top} - T_{bot}} - \tilde{z} \tag{1.12}$$

Now, we can nondimensionalize the equations (1.5)-(1.7)

$$\nabla \cdot \tilde{u} = 0 \tag{1.13}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \nabla \tilde{u} = -\nabla \tilde{p} + \frac{\nu \nabla^2 \tilde{u}}{\kappa} + \frac{\nu \alpha g (T_{bot} - T_{top}) d^3}{\nu \kappa^2} \tilde{\Theta} \hat{z} = -\nabla \tilde{p} + \frac{\nu}{\kappa} \nabla^2 \tilde{u} + \frac{\nu}{\kappa} \frac{\alpha g (T_{bot} - T_{top}) d^3}{\nu \kappa} \tilde{\Theta} \hat{z} \tag{1.14}$$

$$\frac{\partial \tilde{\Theta}}{\partial \tilde{t}} + \tilde{u} \cdot \nabla \tilde{\Theta} = w + \nabla^2 \tilde{\Theta} \tag{1.15}$$

In (1.14) we have the number $\frac{\nu}{\kappa}$. This is the so called Prandl number which we denote by $Pr = \frac{\nu}{\kappa}$. We also have the so called Rayleigh number Ra , given by the equation.

$$Ra = \frac{\alpha g (T_{bot} - T_{top}) d^3}{\nu \kappa} \tag{1.16}$$

We denote by (u, v, w) the velocity components in the x,y and z directions. Beause of the complexity of the calculation that would follow in the next steps on the way to derive the Lorenz Equations we just give the ideas that lead us to our goal. We now consider the convection in the form of two-dimensional convection rolls, say aligned with the x-axis.Then we have to work with the Truncated Galerkin Expansion, that will lead us to the Lorenz System.

2 Mathematical description

2.1 Simple properties of the Lorenz System

In this section we'll try to eliminate all possibilities for the long-term behavior of the following system:

$$\dot{x} = \sigma(y - x) \tag{2.17}$$

$$\dot{y} = rx - y - xz \tag{2.18}$$

$$\dot{z} = xy - bz \tag{2.19}$$

In this equation we have 3 parameters: $\sigma, r, b > 0$ [4]

2.1.1 Nonlinearity

It is immediately clear, that the Lorenz system has two nonlinearities, the quadratic terms xy and xz.

2.1.2 Symmetry

In this section, we want to figure out, what happens if we replace(x, y) with $(-x, -y)$ in (2.1)-(2.3).So let us put $(-x, -y)$into the "original" equations.

1. $$\sigma(y - x) \to \sigma(-y - -x) = \sigma(-y + x) \tag{2.20}$$

2. $$rx - y - xz \to \; -rx + y + xz \tag{2.21}$$

3. $$xy - bz \to xy - bz \tag{2.22}$$

It is obvious, that the equations stay the same, which means, we have an interesting symmetry in the Lorenz equations. For the solution $(x(t), y(t), z(t))$ we find the symmetric partner$(-x(t), -y(t), z(t))$

2.1.3 Fixed points

We want to determine the equilibrium points of the Lorenz system, given by the equations:

$$\dot{x} = \sigma(y - x) \tag{2.23}$$

$$\dot{y} = rx - y - xz \tag{2.24}$$

$$\dot{z} = xy - bz \tag{2.25}$$

The origin

$$O = (0, 0, 0) \tag{2.26}$$

[4] σ is the so called Prandtl number,which is a dimensionless number, named after the german scientist Ludwig Prandtl. It is defined as the ratio of kinematic viscosity to thermal diffusivity
$\mathring{}$ is the Rayleigh number is also a dimensionless number, named after Lord Rayleigh . It is defined as the product of the Grashof number and the Prandtl number

is of course an equilibrium point
We have to make the following transformations:

$$0 = \sigma(y - x) \tag{2.27}$$

$$0 = x - y \tag{2.28}$$

$$x = y \tag{2.29}$$

Now, we take a look to (2.18)

$$0 = rx - y - xz \tag{2.30}$$

We know from (2.29) that $x = y$, which means, we can rewrite this equation in the following form

$$0 = rx - x - xz \Rightarrow x(r - 1 - z) \tag{2.31}$$

$$0 = r - 1 - z \Rightarrow z = r - 1 \tag{2.32}$$

Which means, we have the z-coordinate of the equilibrium point Now, we can calculate the other two coordinates. If we take a closer look at (2.19),we find

$$0 = xy - bz \tag{2.33}$$

by using (2.19)and(2.29) we determine

$$0 = x^2 - b(r - 1) \tag{2.34}$$

Mutatis mutandis

$$x = \pm\sqrt{b(r - 1)} \tag{2.35}$$

On the same way, we can also calculate the y-coodrinate

$$y = \pm\sqrt{b(r - 1)} \tag{2.36}$$

After all these transformations we have determinded the eqilibrium points

$$Q^+ = (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, r - 1) \tag{2.37}$$

$$Q^- = (-\sqrt{b(r - 1)}, -\sqrt{b(r - 1)}, r - 1) \tag{2.38}$$

$$O = (0, 0, 0) \tag{2.39}$$

Q^+ and Q^- stand for the left- or right turning convention rolls. These two equilibria only exist if $r > 1$, which means, we have a bifurcation for $r = 1$.

2.2 Stability

2.2.1 Linear stability of the origin

First thing we have to do in this section is the linearization of the system at the origin. It is very easy to do , we just omit the nonlinearities of ((2.17)-(2.19)). Then we have $\dot{x} = \sigma(y - x)$,$\dot{y} = rx - y$ and $\dot{z} = -bz$.
Another way, to get the same solution is to say that $x = y = 0$. Having this situation implicates $\dot{x} = \dot{y} = 0$. If this is the case, we have to deal with,we are very lucky, because it is equivalent with the statement: the z-axis is invariant. That means, we have $\dot{z} = -bz$, so all the solutions on this axis tend to the origin. This is the reason,why we can say, it suffices to view the following matrix

$$A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Now ,we want to calculate the **eigenvalues** of this linear map. The way, we are working with is the usual one. All we have to do is to calculate the **determinant of the characteristic polynomial p(λ)**.

$$p(\lambda) = det \begin{pmatrix} -\sigma - \lambda & \sigma \\ r & -1 - \lambda \end{pmatrix}$$

The eigenvalues of the linear map are the roots of this quadratic polynomial, which are easy to calculate analytically

$$p(x) = (-\sigma - \lambda)(-1 - \lambda) - r\sigma = \sigma + \sigma\lambda + \lambda + \lambda^2 - \sigma r = \lambda^2 + \lambda(\sigma + 1) + \sigma(1 - r) \quad (2.40)$$

$$\lambda_{1,2} = \frac{(\sigma + 1)}{2} \pm \sqrt{(\frac{(\sigma + 1)}{2})^2 - \sigma(1 - r)} \quad (2.41)$$

$$\lambda_{1,2} = \frac{(\sigma + 1)}{2} \pm \sqrt{\frac{(\sigma + 1)^2}{4} - \frac{4\sigma(1 - r)}{4}} \quad (2.42)$$

$$\lambda_{1,2} = \frac{1}{2}((\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}) \quad (2.43)$$

Note, that both eigenvalues are negative, when $0 \leq r < 1$. $\rightarrow$ The origin is a sink in this case.

Now, we want to prove the following

Proposition:
For $r < 1$ all solutions of the Lorenz system tend to the equilibruim point at the origin

Proof. Our first step by proving this equation is to construct a Lyapunov function [5]

$$L(x, y, z,) = x^2 + \sigma y^2 + \sigma z^2 \tag{2.44}$$

Now, we derivate

$$\dot{L} = 2x\dot{x} + 2\sigma\dot{y} + 2\sigma\dot{z} \tag{2.45}$$

Using the chain rule and the values $\dot{x}, \dot{y}, \dot{z}$ from (2.17)-(2.19), we have :

$$\dot{L} = 2x\sigma(y - x) + 2\sigma y(rx - y - xz) + 2z\sigma(xy - bz) \tag{2.46}$$

$$\dot{L} = 2\sigma(x(y - x) + y(rx - y - xz) + z(xy - bz)) \tag{2.47}$$

$$\dot{L} = 2\sigma(xy - x^2 + rxy - y^2 - xyz + xyz - bz^2) \tag{2.48}$$

$$\dot{L} = 2\sigma(-x^2 - y^2 + xy(r + 1) - bz^2) \tag{2.49}$$

$$\dot{L} = -2\sigma(x^2 + y^2 - xy(r + 1) + bz^2) \tag{2.50}$$

$$\dot{L} = -2\sigma(x^2 + y^2 - xy(r + 1)) - 2\sigma(bz^2) \tag{2.51}$$

It is obvious that $\dot{L} < 0$. Now let us define a function $g(x, y)$

$$g(x, y) = x^2 + y^2 - xy(r + 1) \tag{2.52}$$

We see , that $g(x, y) = x^2 + y^2 - xy(r + 1) > 0$ for all points $(x, y) \neq 0$. This statement is obviously true if we move along the y-axis. Now we want to answer the question: What happens if we have another straight line $y = kx$ in the xy-plane? Clearly, we have to write:

$$g(x, kx) = x^2 + (kx)^2 - xkx(r - 1) \tag{2.53}$$

Mutatis mutandis

$$g(x, kx) = x^2(1 + k^2 - k(r - 1)) \tag{2.54}$$

It is clear,that for $r < 1$ the term $x^2(1 + k^2 - k(r - 1))$ is positive $\Rightarrow$ This is exactely what we wanted to show □

2.2.2 Global stability of the origin

Now, we are moving on in our studies to the Lorenz system and want to prove the following

[5] We consider nonlinear time-invariant system $\dot{x} = f(x)$,where $f : \mathbb{R}^n \to \mathbb{R}^n$ and x^* an equilibrium point of $f(x)$ (if $f(x^*) = 0$).Suppose, that we can find a Lyapunov function $L(x)$ (continuos, differentiable, real valued), with following properties:

$L(x) > 0$ for all $x \neq x^*$
$\dot{L} < 0$ for all $x \neq x^*$

Proposition:
The equilibrium points $Q_{\pm}$ are sinks provided

$$1 < r < r^* = \sigma(\frac{\sigma + b + 3}{\sigma - b - 1}) \tag{2.55}$$

Proof. From the linearization, we have the matrix

$$B = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{pmatrix}$$

Now, we calculate the eigenvalues at $Q_{\pm}$,which means ,we put first $x = \sqrt{b(r-1)}$,$y = \sqrt{b(r-1)}$, $z = r - 1$, than $x = -\sqrt{b(r-1)}$,$y = -\sqrt{b(r-1)}$, $z = r - 1$ into the matrix B and calculate the characteristic polynomial $p_r(\lambda)$

$$p_r(\lambda) = det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ (r - (r-1)) & -1 - \lambda & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b - \lambda \end{pmatrix}$$

$$p_r(\lambda) = det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ r+1 & -1 - \lambda & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b - \lambda \end{pmatrix}$$

$= ((-\sigma - \lambda)(-1 - \lambda)(-b - \lambda))+((\sqrt{b(r-1)})(\sigma)(-\sqrt{b(r-1)})$-
$((\sqrt{b(r-1)})(-\sqrt{b(r-1)})(-\sigma - \lambda))$ -
$((-b - \lambda)(r+1)(\sigma))$
$=$
$(-\sigma b - \sigma\lambda - \sigma b\lambda - \sigma\lambda^2 - \lambda b - \lambda^2 - b\lambda^2 - \lambda^3 - b(r-1)\sigma - b(r-1)\sigma - b(r-1)\lambda + br\sigma + b\sigma + \lambda r\sigma + \lambda\sigma)$
$=(-\lambda^3 - \lambda^2(1 + b + \sigma) - 2b\sigma(r-1) - b\lambda(r+\sigma))$ [6]

So, we have:

$$p_r(\lambda) = -\lambda^3 - \lambda^2(1 + b + \sigma)) - 2b\sigma(r-1) - b\lambda(r+\sigma) \tag{2.56}$$

As usual, we are looking for the roots of $p_r(\lambda)$ First, we want to see, what happens, when $r = 1$
For $r = 1$ we have:

$p_1(\lambda) = -\lambda^3 - \lambda^2(1 + b + \sigma) - b\lambda(1 + \sigma)$ As already mentioned, we are looking for the roots of this equation

$$0 = -\lambda^3 - \lambda^2(1 + b + \sigma) - b\lambda(1 + \sigma) \tag{2.57}$$

$$0 = \lambda^3 + \lambda^2(1 + b + \sigma) + b\lambda(1 + \sigma) \tag{2.58}$$

$$0 = \lambda(\lambda^2 + \lambda(1 + b + \sigma) + b(1 + \sigma)) \tag{2.59}$$

$$\lambda_{1,2} = \frac{-(1 + b + \sigma) \pm \sqrt{(1 + b + \sigma)^2 - 4(b(\sigma + 1))}}{2} \tag{2.60}$$

[6] For the calculation of the determinant of B, we used the Sarrus formula

$$\lambda_{1,2} = \frac{-(1+b+\sigma) \pm \sqrt{1+b^2+\sigma^2+2\sigma-2b-2b\sigma}}{2} \tag{2.61}$$

$$\lambda_{1,2} = \frac{-(1+b+\sigma) \pm \sqrt{(1-b+\sigma)^2}}{2} \tag{2.62}$$

$$\lambda_1 = \frac{-(1+b+\sigma)+(1-b+\sigma)}{2} = \frac{-2b}{2} = -b \tag{2.63}$$

$$\lambda_2 = \frac{-(1+b+\sigma)-(1-b+\sigma)}{2} = \frac{-2\sigma-2}{2} = -\sigma-1 \tag{2.64}$$

It is evident that $\lambda_3 = 0$ is also a root. After all this calculation we can say that $p_1(\lambda)$ has distinct roots at $\lambda_1 = -b, \lambda_2 = -\sigma-1$ and$\lambda_3 = 0$. These roots are actually distinct since $\sigma > b+1$, so we have:

$$\sigma > b+1 \Rightarrow -\sigma < -b-1 \Rightarrow -\sigma+1 < -b \tag{2.65}$$

$$-\sigma-1 < -\sigma+1 < -b < 0 \tag{2.66}$$

For values r, that are close to but greater than 1, $p_r(\lambda)$ has three real roots. All roots **are close** to these values.If we take a look at the graph of $p_r(\lambda)$, we realize, that for r close to 1,all roots must be real and negative.

Don't forget:$p_r(\lambda) > 0$ *for all* $\lambda \geq 0$*!!!!*

Now, we have been analysing long enough ,what happens for r close to 1. We also want to know, what happens, when r increases.So,we let r increase. The question, we want to find the answer for is:

What is the lowest value of r, for which $p_r(\lambda)$ has an eigenvalue with zero real part ?

It is clear that the eigenvalue, we are looking for, must have the form $\pm i\eta$ with $\eta \neq 0$. We have already mentioned, that $p_r(\lambda)$ is a real function, which means, it has no roots equal to zero for $r > 0$. Now, we just have to solve $p_r(i\eta) = 0$. We just have to equate both real ($\Re$) and imaginary ($\Im$) parts to zero.
Let us define following variables $a_2 = 1+b+\sigma$, $a_1 = 2b\sigma(r-1)$, $a_0 = b(r+\sigma)$.
$p_r(i\eta) = 0 \Rightarrow$ -$p_r(i\eta) = 0 \Rightarrow -i\eta^3 - a_2\eta^2 + a_1 i\eta + a_0 = 0$. Now we use the method of equating the coefficients.
$\Rightarrow 0 = i\eta(\eta^2 - a_1) \wedge 0 = a_0 - a_2\eta^2 \Rightarrow 0 = \eta^2 - a_1 \wedge 0 = a_0 - a_2\eta^2 \Rightarrow 0 = a_0 - a_2 a_1 \Rightarrow$
$r^* = \sigma(\frac{\sigma+b+3}{\sigma-b-1})$. □

Now we summarize what we already know

1. $0 < r < 1$ the origin $(0,0,0)$ is globally stable, all trajectories are attracted to it.
2. In case $r = 1$, we have a so called pitchfork bifurcation. We have two critical points Q^+ and Q^-, the origin loses its stability.
3. $0 < r < r^*$: Q^+ and Q^- are stabel, critical points
4. We imagine, that Q^+ and Q^- are surrounded by a small, stable limit circle. This occurs by a supercritical Hopf bifurcation. So the fixed points are stable, encircled by a so called saddle cycle [7] If r get closer to r^*, our saddle cycle shrinks down around the fixed point. At $r = r^*$ the fixed points Q^+ and Q^- change into a saddle point
So in case $r = r^*$: Q^+ and Q^- lose their stability in a Hopf bifurcation

[7] A saddle cycle is an unstable limit cycle, that is possible only in phase spaces of three or more dimensions (See: [2])

5. We have to answer the question: What happens if $r > r^*$?
For $r > r^*$ there are clearly no nearby attractors, so the trajectories must change to an other attractor.
Lorenz could show, that the trajectories have for $r > r^*$ an extraordinary long-term behavior. Like balls in a pinball machine, they are repelled from one unstable object after another. At the same time, they are confined to a bounded set of zero volume, yet their manage to move on this set forever without intersecting themselves or others [2].

Proposition

There exists v^* such that any solution that starts outside the ellipsoid $V = v^*$ eventually enter sin this ellipsoid and then remains trapped therein for all future time.

Proof. Let $x = (x, y, z) \in \mathbb{R}^3$ We consider the equation

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \tag{2.67}$$

V defines an ellipsoid centered at (0,0,2r) and is a Lyapunov function

We compute:

$$\dot{V} = 2rx\dot{x} + 2y\sigma\dot{y} + 2\sigma(z - 2r)\dot{z}^8 \tag{2.68}$$

$$\dot{V} = 2rx(\sigma(y - x)) + 2y\sigma(rx - y - xz) + 2\sigma(z - 2r)(xy - bz) \tag{2.69}$$

$$\dot{V} = 2\sigma(rx(y - x) + y(rx - y - xz) + (z - 2r)(xy - bz) \tag{2.70}$$

$$\dot{V} = 2\sigma(rxy - rx^2 + rxy - y^2 - xyz + xyz - bz^2 - 2rxy + 2rbz \tag{2.71}$$

$$\dot{V} = 2\sigma(-rx^2 - y^2 - b(z^2 - 2rz)) \tag{2.72}$$

$$\dot{V} = -2\sigma(rx^2 + y^2 + b(z^2 - 2rz)) \tag{2.73}$$

$$\dot{V} = -2\sigma(rx^2 + y^2 + b(z - r)^2 - br2) \tag{2.74}$$

It is clear, that

$$\vartheta = rx^2 + y^2 + b(z - r)^2 \tag{2.75}$$

defines an ellipsoid,when $\vartheta > 0$. For $\dot{V} < 0$ we should have $\vartheta > br^2$. That means, we have to choose v^* large enough so that the ellipsoid $V = v^*$ **strictly contains** $rx^2 + y^2 + b(z - r)^2 = br^2$. In this case, we have $\dot{V} < 0$ for all $v \geq v^*$. □

The reader of this article might have the question: Why is it important to make all these calculations? The consequence of the last proposition is, that **all solutions** , starting far from the origin **are attracted** to a set that sits inside the ellipsoid $V = v^*$ We denote the set of the points whose solutions remain for all time in $V = v^*$ with Λ.

[8] We use the chain rule and $\dot{x}, \dot{y}, \dot{z}$ from (2.1)-(2.3)

2.3 Volume contraction

In this section, we are going to study the volume of the set Λ. For our following calculations is it indispensable to repeat the definiton on the divergence of a vector field $F(x)$ on the space $\mathbb{R}^3$

$$divF = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}(X) \tag{2.76}$$

The divergence of F measures the speed of the volume changing under the flow Φ_t of F. Now we let G be a region in $\mathbb{R}^3$ with a smmoth boundary and let $G(t)$ be the image of G under the time t map on the flow ($=\Phi_t$). We denote the volume of $G(t)$ with $V(t)$. Using Liouville's theorem, we have the equation:

$$\frac{dV}{dt} = \int_{G(t)} divF dxdydz \tag{2.77}$$

For the Lorenz system (L) we can easily calculate the divergence

$$divL = \frac{\partial}{\partial x}(\sigma(y-x) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) = -\sigma - 1 - b \tag{2.78}$$

Since $divL = -\sigma - 1 - b$ is constant we can write:

$$\dot{V} = \frac{dV}{dt} = \int_{G(t)} (divL) dxdydz = \int_{G(t)} (divL) dV = \int_{G(t)} (-\sigma - 1 - b) dV = -(\sigma + 1 + b)V \tag{2.79}$$

Solving this differential equation leads us to

$$V(t) = V(0)e^{-(\sigma+1+b)t} \tag{2.80}$$

⇒ Volumes in phase space shrink exponentially fast to 0. We already know that **all solutions** , starting far from the origin **are attracted** to a set that sits inside the ellipsoid $V = v^*$ and we call this set Λ. Considering the equation (2.67)asserts that**the volume of Λ is 0.**

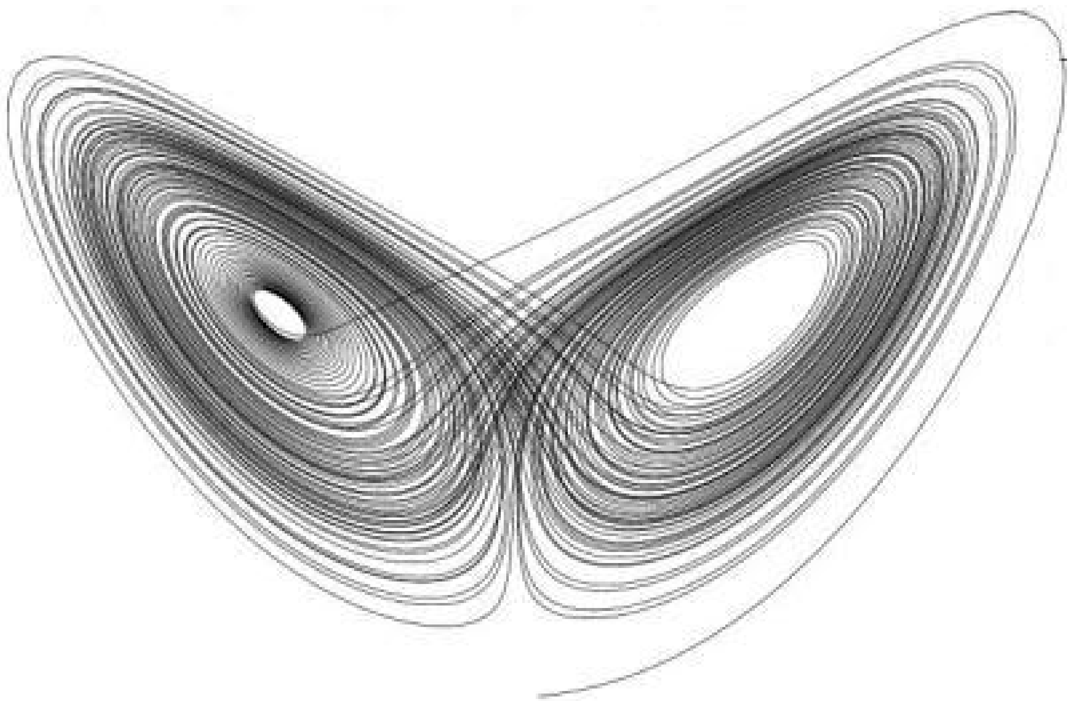

Figure 2: The Lorenz Attractor

2.4 Chaos

2.4.1 Defining Chaos

Before we give a definition for *chaos*[2], we have to mention that there is no generally accepted definition of this phenomenon yet.We try to give a definition that includes all important properties of chaos/chaotic systems.

Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

After the definition, we have to clear what the *ingredients* mean

1. **Aperiodic long-term behavior:**
   We can find trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$

2. **Deterministic:**
   The system has no random inputs or parameters. The irregular behavior arises from the nonlinearity of the system.

3. **Sensitive dependence:**
   Nearby trajectories separate exponentially fast ($\Longleftrightarrow$ tha system has positive Lyapunov exponent)

2.5 Attractor

It is not easy to find a definition of attractor. To give a definiton that can be acceptable for all readers of this article we mix the definitions given in [2] and [4]

2.5.1 Defining Attractor and Strange Attractor

Definition (Attractor):
A closed set Λ is called an attractor, if following properties hold:

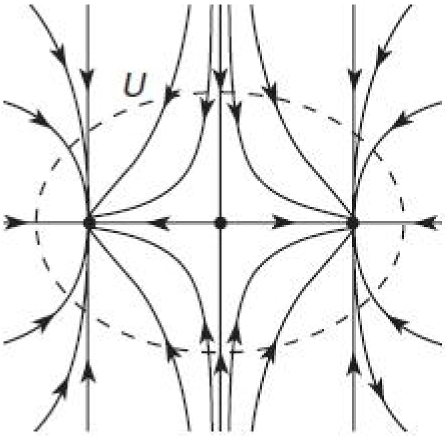

Figure 3: The interval on the x-axis between the two sinks is not an attractor for this system

1. Λ is invariant : any trajectory $x(t)$ that starts in Λ , stays in Λ for $t \to \infty$
2. Λ attracts an open set of initial conditions: there is an open set U containing Λ such that if $x(0) \in U$, then the distance from $x(t)$ to Λ tends to zero as $t \to \infty$. This means that Λ attracts all trajectories that start sufficiently close to it

   $$\Longleftrightarrow$$

   Λ is transitive: Given any points $Y_1, Y_2 \in \Lambda$ and any open neighborhoods U_j about Y_j in U, there is a solution curve that begins in U_1 and later passes through U_2 .
3. Λ is minimal : There is no proper subset of Λ that satisfies conditions 1 and 2.

Now, we want to give an example to understand, what we exactelly mean by an attractor and to show, that all the properties we gave play an important role.

Example

We consider the so called planar system:

$$\dot{x} = x - x^3 \tag{2.81}$$

$$\dot{y} = -y \tag{2.82}$$

We denote the interval $-1 \leq x \leq 1, y = 0$ by I. We want to show that I is an attractor, which means, we show that the properties 1-3 are satisfied. Now we take a look to the phase portrait. It is obvious that the points $I_1 = (-1, 0)$ and $I_2 = (-1, 0)$ are stable fixed points and the origin $0 = (0, 0)$ is a saddle point. We can just see, that all trajectories that start in I, stays in I for $t \to \infty \Rightarrow$ **Λ is invariant.**

It is also pretty much obvious, that I attracts all trajectories in the xy plane

$$\iff$$

I attracts an open set of initial conditions $\Rightarrow$ property 2 is satisfied

Condition 3 is violated, cause the stable fixed points I_1 and I_2 of the system are proper subsets of I and satisfy the properties 1-2. Which means we have no attractor in case of the planar system, cause the third property is not satisfied.

Definition (Strange Attractor):

A strange attractor is an attractor, that exhibits sensitive dependence on initial conditions.

2.5.2 Chaos on a Strange Attractor

In the last part of this section we study the long-time behavior of the Lorenz system using the parameters $\sigma = 10, b = \frac{8}{3}$ and $r = 28$.These are the original values, Lorenz used by his examinations. This value of r is just past the Hopf bifurcation $r^* = \sigma(\frac{\sigma+b+3}{\sigma-b-1}) \approx 24,74$, that means we have to deal with something strange.

$$B = \begin{pmatrix} -10 & 10 & 0 \\ (28-z) & -1 & -x \\ y & x & -\frac{8}{3} \end{pmatrix}$$

Now we consider the linearization at the origin and calculate the eigenvalues

$$p_\lambda = det \begin{pmatrix} -10-\lambda & 10 & 0 \\ 28 & -1-\lambda & 0 \\ 0 & 0 & -\frac{8}{3}-\lambda \end{pmatrix}$$

$$P_\lambda = (-10-\lambda)(-1-\lambda)(-\frac{8}{3}-\lambda)-(-\frac{8}{3}-\lambda)(28)\cdot(10) = (-\frac{8}{3}-\lambda)(\lambda^2+11\lambda-270) \quad (2.83)$$

The eigenvalues are the roots of the characteristic polynomial

$$(-\frac{8}{3}-\lambda)(\lambda^2+11\lambda-270) = 0 \quad (2.84)$$

$$(-\frac{8}{3}-\lambda) = 0 \Rightarrow -\lambda = \frac{8}{3} \Rightarrow \lambda_1 = -\frac{8}{3} \quad (2.85)$$

$$\lambda^2 + 11\lambda - 270 = 0 \quad (2.86)$$

$$\lambda_{2,3} = \frac{11}{2} \pm \sqrt{(\frac{11}{2})^2 + 270} \Rightarrow \lambda_{2,3} = -\frac{11}{2} \pm \sqrt{\frac{121}{4} + \frac{1080}{4}} \quad (2.87)$$

$$\lambda_{2,3} = \frac{11}{2} \pm \sqrt{\frac{1201}{4}} \Rightarrow \lambda_{2,3} = \frac{11}{2} \pm \frac{\sqrt{1201}}{2} \quad (2.88)$$

So we have the eigenvalues $\lambda_1 = -\frac{8}{3}$ $\lambda_2 = \frac{11}{2} + \frac{\sqrt{1201}}{2}$ and $\lambda_3 = \frac{11}{2} - \frac{\sqrt{1201}}{2}$

We want to know the equilibrium points of the system if we use the original parameter values. To calculate the equilibria we just have to put the parameter values in the coordinates of the equilibrium points we have already calculated in section (2.1.3).

$$Q^+ = (\sqrt{\frac{8}{3}(28-1)}, \sqrt{\frac{8}{3}(28-1)}, 28-1) \quad (2.89)$$

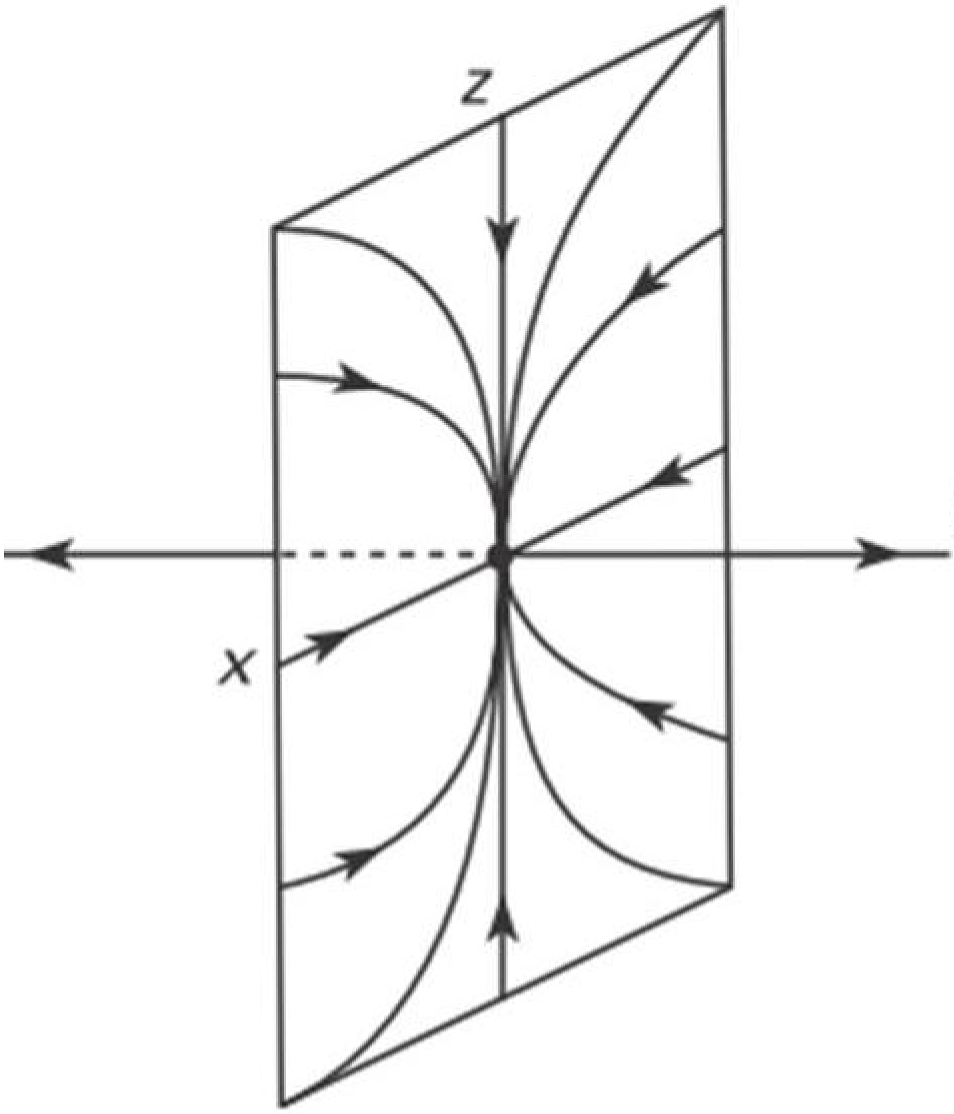

Figure 4: Phase portrait of the linearisation at the origin

$$Q^- = (-\sqrt{\frac{8}{3}(28-1)}, -\sqrt{\frac{8}{3}(28-1)}, 28-1) \tag{2.90}$$

$$O = (0,0,0) \tag{2.91}$$

So we have the equilibria : $Q^- = (-\sqrt{72}, -\sqrt{72}, 27)$, $Q^+ = (\sqrt{72}, \sqrt{72}, 27)$ and O=(0,0,0)
So we can give the linearized system at the origin by the matrix

$$A = \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} X$$

If we take a look at the phase portrait,shown in Figure 4 we can see,that all solutions in the stable plane of this system tend to the origin tangentially to the z-axis. After all these calculations Lorenz began to integrate from initial condition $(0; 1; 0)$, close to the saddle point at the origin. After an initial transient the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly[2], which is equivalent with the statement, that the motion,he observed is aperiodic. He also obeserved that we get a very beautiful structure if we plot $x(t)$ against $z(t)$ (See Figure 2.). Looking at the trajectory, we could believe, that the trajectory cross itself repeatedly, but that is not

true. That is just an artifact of projecting the three-dimensional trajectory onto a two dimensional plane.
We can plot this with mathematica if we use following code:

$$\begin{gathered}
\text{In}[1] := \sigma = 10; r = 28; b = 8/3; \\
\text{sol} = \text{NDSolve}[\{ \\
x'[t] == -\sigma(x[t] - y[t]), \\
y'[t] == -x[t]z[t] + rx[t] - y[t], \\
z'[t] == x[t]y[t] - bz[t], \\
x[0] == 30,\ y[0] == 10,\ z[0] == 40\ \}, \\
\{x, y, z\}, \{t, 0, 20\}, MaxSteps \rightarrow 5000]; \\
ParametricPlot3D[Evaluate[\{x[t], y[t], z[t]\}/.sol], \{t, 0, 20\}, \\
\text{PlotPoints}\rightarrow 2000,\ \text{Axes} \rightarrow \text{False},\ \text{PlotRange} \rightarrow \text{All}];
\end{gathered}$$

Let us make the following steps to understand the trajectory ,shown in Figure 2., in detail:

1. The trajectory starts near to the origin , then swings to the right
2. Diving into the center of a spiral on the left
3. After a very slow spiral outward, shooting back over to the right side
4. Spiraling around a few time and shooting back over to the left side and spiraling around
5. Repeating this behaviour infinitely

The number of circuits made on either sides varies unpredictably from one cycle to the next [2].
We can also study this trajectory in all three dimensions. It can be more useful than looking at the two dimensional projection only. So viewing at the 3D system appears to settle onto an exquisitely thin set that looks like a pair of butterfly swings. For a more detailed description take a look into [2]

Exponential divergence of Nearby Trajectories

The motion on the attractor exhibits sensitive dependence on initial conditions $\Rightarrow$ Two trajectories starting very close together will rapidly diverge from each other. That means for us that long-term prediction becomes impossible in a system like this, where uncertainties are amplified enormously fast. We let transients decay, so that a trajectory is on the arrtactor. We denote by $x(t)$ the point on the attractor at time t. Now we pick a nearby point, say $x(t) + \rho(t)$, where ρ is a very small separation vector of initial lenght $\|\rho_0 = 10^{-15}\|$. The only thing we have to do is to watch, how $\rho(t)$ grows. After a lot of time passed, we can observ that $\|\rho(t)\| \sim \|\rho_0\| e^{\lambda t}$, where $\lambda = 0.9 \Rightarrow$ Neighboring trajectories separate exponentially fast.
Plotting $ln\|\rho(t)\|$ versus t lead us to the same result. We find a curve that is close to a straight line with a positive slope λ (See Figure 5)

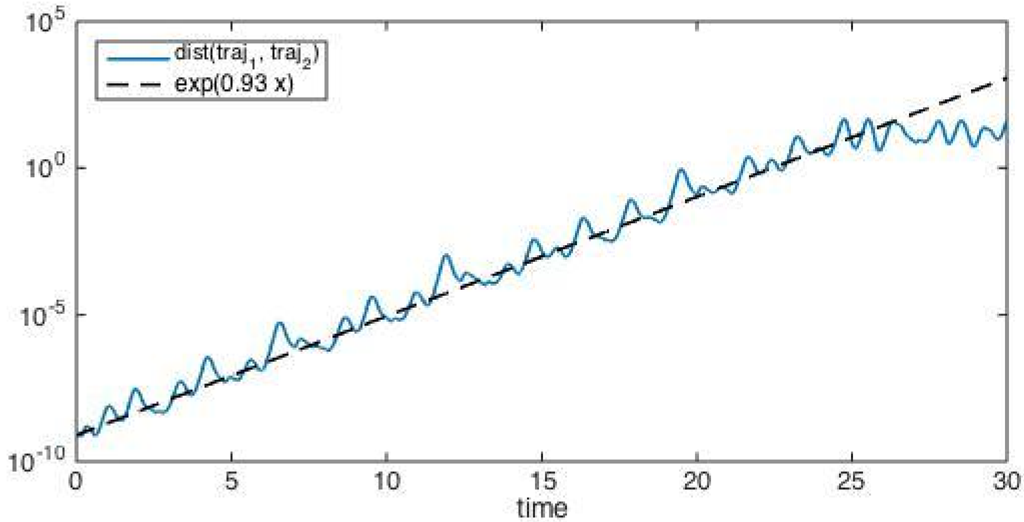

Figure 5: Curve and straight line with $\lambda = 0.93$

There are a few things we have to make clear:

1. The curve is never exactely straight
2. The exponential divergence must stop when the separation is comparable to the diameter of the attractor
3. λ is the so called Lyapunov exponent

When $\lambda > 0$, there is a time horizon beyond which prediction breaks down. We can try to measure the initial condition of an experimential system very precise. What ever, there will always be some errer $\|\rho_0\|$ between our estimate and the true initial state. After a time t, our discrepancy grows to $\|\rho(t)\| \sim \|\rho_0\|e^{\lambda t}$. Now, let us denote the measure of our tolerance by a. If a prediction is within a of the true state, we consider it acceptable. Our prediction becomes intolerable for $\|\rho_t\| \geq a$. This occurs after a time:

$$t_{horizon} \sim O(\frac{1}{\lambda} ln \frac{a}{\|\rho_0\|}) \tag{2.92}$$

Even if we are working very hard, we can not reduce the initial measurement error, we can't predict longer than a few multiples of $\frac{1}{\lambda}$

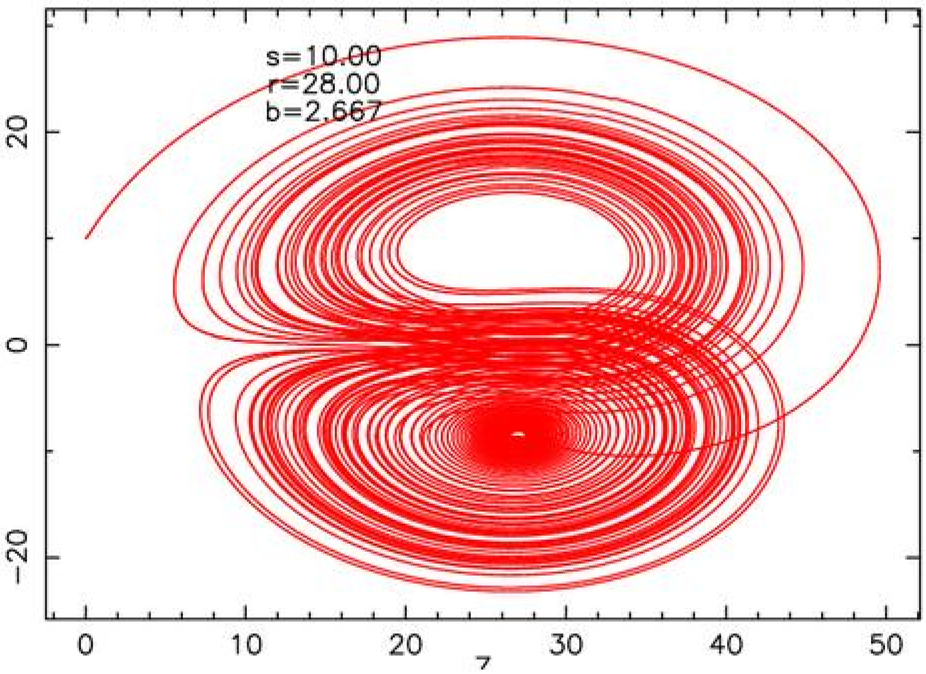

Figure 6: Phase portrait in the y-z plane

3 The Lorenz Map

There are many ways to analyse the dynamics on Lorenz's strange attractor, but Lorenz himself might found the most beautiful one. His really simple idea was to direct our attention to a particular view of the attractor (Figure 4). Now, we are going to cite from his own notes:

the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit

Lorenz's idea was that he considers the function $z(t)$ and looks at its maxima. He denoted the nth local maximum of $z(t)$ with z_n. He thought that z_n is able to predict z_{n+1}. To check this he used numercal integration again. He integrated the function $z(t)$ and measured its local maxima. Finally he plotted z_{n+1} against z_n (see Figure 5).
The data from the chaotic time series appear to fall neatly on a curve.
By this trick Lorenz was able to extract order from chaos. The function $z_{n+1} = f(z_n)$ is the so called Lorenz map. It is very useful if we want to study the dynamics on the attractor. It is clear thet the Lorenz map is an iterated map, cause we have $z_1 = f(z_0)$, $z_2 = f(z_1)$, $z_3 = f(z_2)$ and so on.
Now we have make clear that the function ,shown in Figure 5 is not exactely a curve. $f(z)$ is not well defined, cause there can exist more than one output z_{n+1} for a given input z_n.

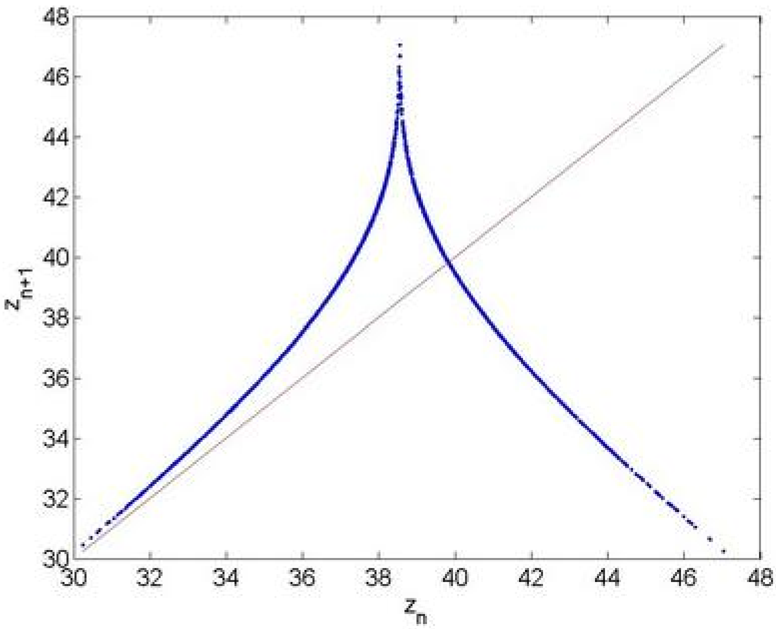

Figure 7: z_{n+1} vs. z_n

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